On the Number of Prime Factors of Square-Free Integers

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Every integer greater than 1 can be expressed as a unique product of prime factors (Fundamental Theorem of Arithmetic).

A *square-free* integer is one that is not divisible by any perfect square. It follows that each of its prime factors occurs only once.

For convenience, we classify the integer 1 as square-free, and we note that it has an even number of prime factors, namely zero.

Consider the set of square-free integers for which the highest prime factor is the kth prime, p_k. Denote this set by $S(k)$.

Each integer in $S(k)$ corresponds to a subset of the first k primes. Each prime may be either included in or excluded from the set of factors, so there are 2^k possibilities (corresponding to the members of the power set of the first k primes).

Consider also the subset of $S(k)$ containing integers with an even number of prime factors, which we denote by $S_e(k)$, and that containing integers with an odd number of factors, which we denote by $S_o(k)$. Suppose the cardinalities of these subsets are $N_e(k)$ and $N_o(k)$ respectively. Clearly, $N_e(k) + N_o(k) = 2^k$ (because we include the integer 1).

Now we step to the next prime, p_{k+1}. The set $S(k+1)$ can be generated from $S(k)$. Every member of $S(k)$ is a member of $S(k+1)$ which omits p_{k+1}. The elements that include p_{k+1} are generated by multiplying each member of $S(k)$ by p_{k+1}. Thus we generate all 2^{k+1} members.

The set $S_e(k+1)$ includes all the members of $S_e(k)$ and it includes the members of $S_o(k)$ multiplied by p_{k+1}. Thus $N_e(k+1) = N_e(k) + N_o(k)$, and analogously $N_o(k+1) = N_o(k) + N_e(k)$. Hence $N_e(k+1) = N_o(k+1)$.

This argument is valid for all k greater than 1. When $k = 1$, $p_1 = 2$ and $S(1) = \{1,2\}$. The subsets with even and odd numbers of factors are $S_e(1) = \{1\}$ and $S_o(1) = \{2\}$. These subsets are equal in size.

We conclude that, for all k, the set of square-free numbers with maximum prime factor less than or equal to p_k has exactly 2^{k-1} members with an even number of prime factors and 2^{k-1} members with an odd number.

Comment

The members of $S(k)$ are not consecutive integers. The minimum member is, of course, 1 and the maximum member is the product of the first k primes.

Every square-free integer is represented in $S(k)$ for some value of k, so all are covered.